

Witten's $SU(2)$ anomaly on the lattice.

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Abstract

Witten's anomaly for $SU(2)$ with a single $I = \frac{1}{2}$ Weyl fermion in four dimension is shown to be reproduced by the lattice overlap. The mechanism is based on Berry's phase, and on the analyticity of the matrix H by which the overlap is defined.

Although Witten's anomaly (WA) [1] in $SU(2)$ gauge theory cannot be *directly* seen in perturbation theory, (see [2]) it is, conceptually, the simplest example of a chiral anomaly in four dimensions [3]. It seems therefore reasonable to expect any general *non-perturbative* scheme for chiral gauge theories to contain WA in an *evident* way. The first arguments for the presence of WA in the overlap [4,5] (not relying on an embedding of WA in a perturbative anomaly of a larger group [2]) were given in [6]. Further work related to the realization of global anomalies in the overlap was carried out in [7,8,9]. I know of no evidence for WA in other non-perturbative attempts to construct chiral gauge theories. In this paper I follow the line of thought laid down in [8] and [9] to show that any smooth phase choice (what is meant by this will be made clear later on) for the overlap necessarily reproduces WA.

The main object in the overlap is the many body fermion ground state $|v\{U_l\}\rangle$ which depends parametrically on the background made out of all $SU(2)$ link variables U_l . The second quantized fermion system is non-interacting, with hamiltonian $\mathcal{H} = a^\dagger H a$ and the dependence on $\{U_l\}$ comes in through the matrix H sandwiched between the fermionic creation/annihilation operators a^\dagger, a . H is γ_5 times the standard Wilson-Dirac operator with the hopping parameter κ set to κ_0 , somewhere in the range $\frac{1}{2(d-1)} > \kappa_0 > \frac{1}{2d}$ for Euclidean dimension d . For us, $d = 4$. While the exact structure of H is unimportant, what follows crucially and equally depends both on the locality of H and its analyticity in the link variables $\{U_l\}$.

The chiral determinant is given by the “overlap” $\langle v_{\text{ref}} | v\{U_l\} \rangle$ where, in the simplest version, $|v_{\text{ref}}\rangle$ is U_l -independent and defined just as $|v\{U_l\}\rangle$, only now $H = \gamma_5$. For the case that the gauge group is $SU(2)$ and the fermions have $I = \frac{1}{2}$ one can choose a basis where $H\{U_l\}$ is real for all $\{U_l\}$. An explicit basis was written down in [9], but the fact that such a basis exists must have been known to many people. In this basis it becomes evident that $|v\{U_l\}\rangle$ and $|v_{\text{ref}}\rangle$ can be both taken real. Under a gauge transformation $U \rightarrow U^g$ we have $\mathcal{H}\{U_l^g\} = G^\dagger(g)\mathcal{H}\{U_l\}G(g)$ induced by $H\{U_l^g\} = g^\dagger H\{U_l\}g$. $G(g)$ represents g in the fermionic Fock space. WA simply means that a choice of the states $|v\{U_l\}\rangle$ that is smooth in the gauge background (and in κ_0) would result in

$$|v\{U_l^g\}\rangle = -G^\dagger(g)|v\{U_l\}\rangle \quad (1)$$

for a lattice gauge transformation g which is reasonably close to a continuum gauge transformation g that is an element of the non-trivial homotopy class of maps $T^4 \rightarrow SU(2)$. Since $|v_{\text{ref}}\rangle$ obeys $|v_{\text{ref}}\rangle = G^\dagger(g)|v_{\text{ref}}\rangle$ for any gauge transformation, the sign in equation (1) induces a sign switch in the regularized chiral determinant of the overlap as $\{U_l\} \rightarrow \{U_l^g\}$ and hence a violation of gauge invariance. This is the single kind of gauge

violation that can happen in the overlap; the square of the overlap is gauge invariant, and moreover, obeys a simple identity derived in [10]:

$$(< v_{\text{ref}} | v\{U_l\} >)^2 = \det \frac{1+V}{2}, \quad V \equiv \gamma_5 \epsilon(H), \quad (2)$$

where ϵ is the sign function.

Following [1] I consider a path parameterized by $0 < t < 1$ connecting a configuration $\{U_{l,0}\}$ to $\{U_{l,0}^g\}$ for a nontrivial continuum g , suitably latticized (see [9]). The path is required to consist only of lattice gauge configurations that are close to smooth continuum gauge field configurations. (For example, this requirement excludes deforming the lattice g itself to identity, although this is perfectly legal on the lattice. A single-valued phase choice cannot be also smooth along this “forbidden” path.) As a function of t we expect the non-negative [10] quantity $\det \frac{1+V}{2}$ to go through zero a certain number of times, n , which, counting multiplicities, should satisfy $n = 2(\text{mod})4$.

The direct evaluation of $\det \frac{1+V}{2}$ is not easy numerically, and even if it were, we would lose some insight if I proceeded to establish WA that way. Following [7-9] I choose to go about this differently.

The fermions are understood to obey anti-periodic boundary conditions in all four directions on a four torus of equal sides L . I exclude the parts of the link variables that implement these boundary conditions from the link configurations quoted below. Picking $U_{l,0} \equiv \mathbf{1}$ and the g from [9] I construct a curve in the space of Hamiltonian matrices $H\{U_l, \kappa\}$. The curve starts at point A with $U_l = U_{l,0}$ and $\kappa = \kappa_0$ and follows t from $t = 0$ to $t = 1$ at constant κ to point B where $U_l = U_{l,0}^g$. From B the curve follows κ , at fixed gauge background, to $\kappa = 0$ at point C. At $\kappa = 0$ there is no dependence on the gauge field so the curve is taken to proceed, with constant $U_l = U_{l,0}$ this time, increasing κ back to κ_0 and returning to A. Thus, we have constructed a closed path, ABCA in the space of real matrices H . (Note that we only care about H up to its multiplication by a positive real number; actually, we wouldn't even care about any change $H \rightarrow f(H)$ with smooth, monotonic and real f , also satisfying $f(0) = 0$.) By gauge invariance, anything that happens with $\det \frac{1+V}{2}$ along BC is duplicated along CA , so the number of zeros along BCA vanishes mod four. It is important (and trivially true) that $\epsilon(H)$ stays well defined along the path BCA; no “exceptional” configurations in the sense of the overlap are encountered.

Let us assume that no exceptional configurations are encountered along the path AB either; this is not trivial, but can be confirmed numerically. Then, for WA we need $\det \frac{1+V}{2}$ to vanish $n = 2(\text{mod})4$ times along AB. This happens if and only if $|v\{U_l, \kappa\} >$ picks up a

-1 when transported round the loop. The nice thing now is that in order to show the latter it is sufficient to establish that inside the disk spanned by κ, t and bounded by our loop there is exactly one double cone degeneracy point for the ground state of \mathcal{H} [11]. Note that we *do not* require V to be well defined everywhere inside the disk; all we need there is H , and H , unlike V , is known to be analytic both in κ and in t .

The degeneracy in \mathcal{H} can come (generically) only from a degeneracy in H of the double cone type involving two states closest to zero energy, each crossing zero at the same or close-by points. The tip of the double cone does not have to be at zero energy, but in practice it will be reasonably close to zero energy. A degeneracy satisfying these requirements is excluded for $\kappa < \frac{1}{2d}$ since H has an impenetrable gap around zero energy there, for any gauge field. So, we end up having to search the interior of the region $\frac{1}{2d} < \kappa < \kappa_0$, $0 < t < 1$ for conical degeneracies in H involving the lowest two states of H^2 . This is relatively easy numerically, using the variational principle and the locality of H , employing methods developed in [12].

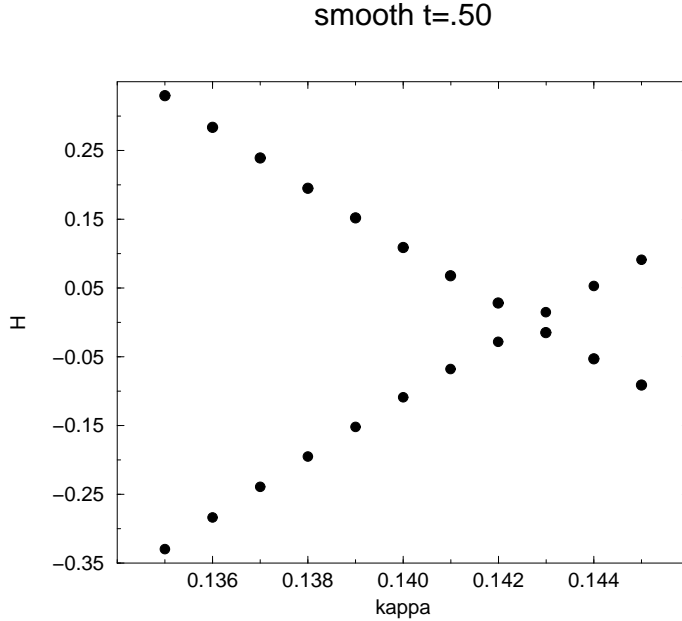


Figure 1. First cross-section through double cone; within numerical accuracy: crossing.

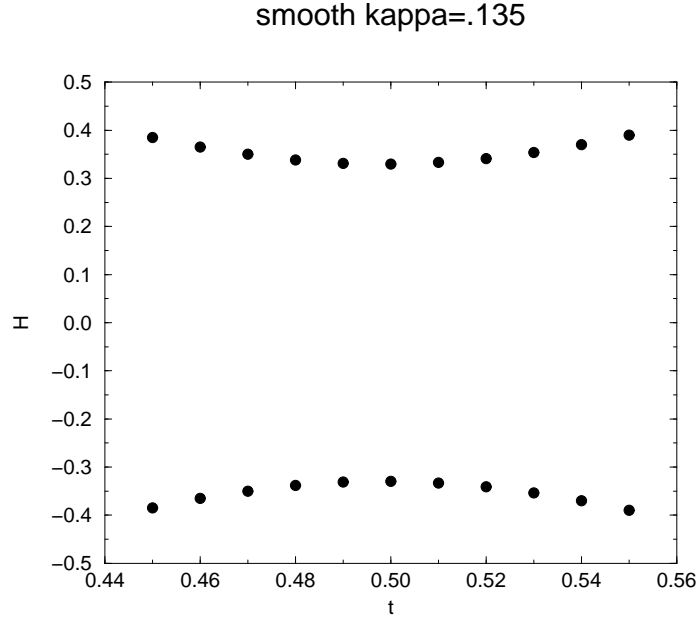


Figure 2. Second cross-section through double cone; avoided crossing.

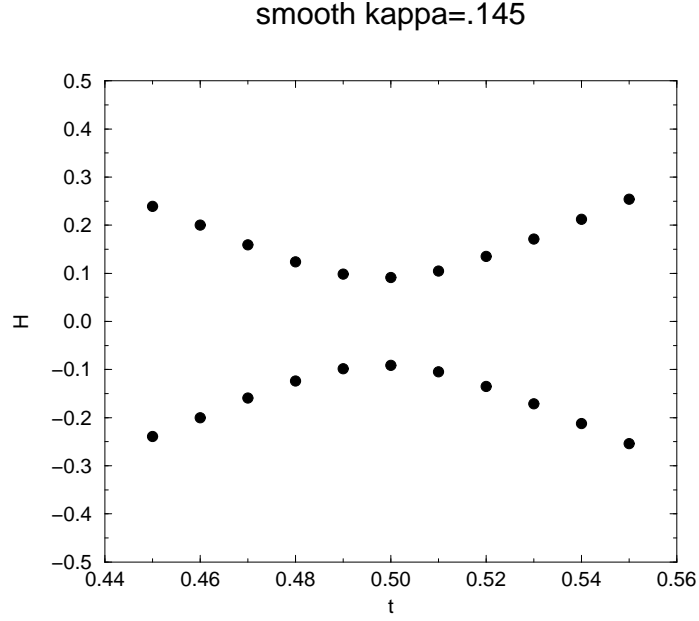


Figure 3. Third cross-section through double cone; avoided crossing.

Working on an L^4 lattice with a linear interpolation between $\{U_{l,0}\}$ and $\{U_{l,0}^g\}$ I searched for the double cone degeneracy in H and found exactly one at $t = .5$ and $\kappa = .1425(5)$ for $L = 8$. Figures 1,2,3 show three cross-sections through the double cone. The lattice gauge configurations were chosen with an accidental symmetry which ensures that the tip of the double cone will be at zero energy exactly. By multiplying g of [9] by unitary

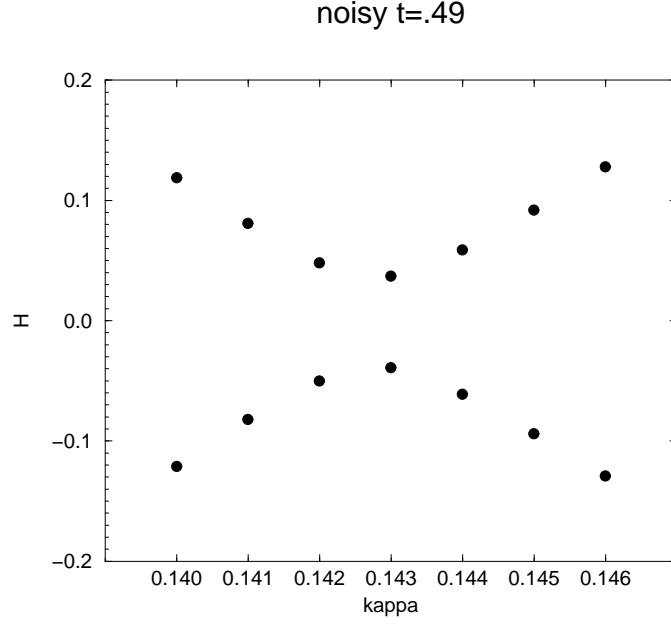


Figure 4. First cross-section through double cone; avoided crossing.

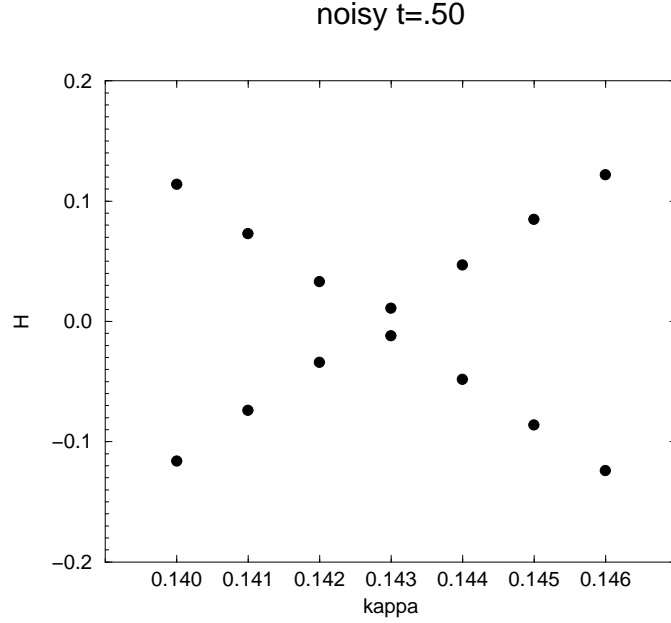


Figure 5. Second cross-section through double cone; within numerical accuracy: crossing.

matrices randomly drawn from a small vicinity of unity, the double cone is made to move slightly, and its tip is no longer exactly at zero, as expected in the generic case. Three cross-sections through the double cone for the noisy gauge background at $L = 8$ are shown in figures 4,5,6. To see how the continuum limit is approached I searched for the double

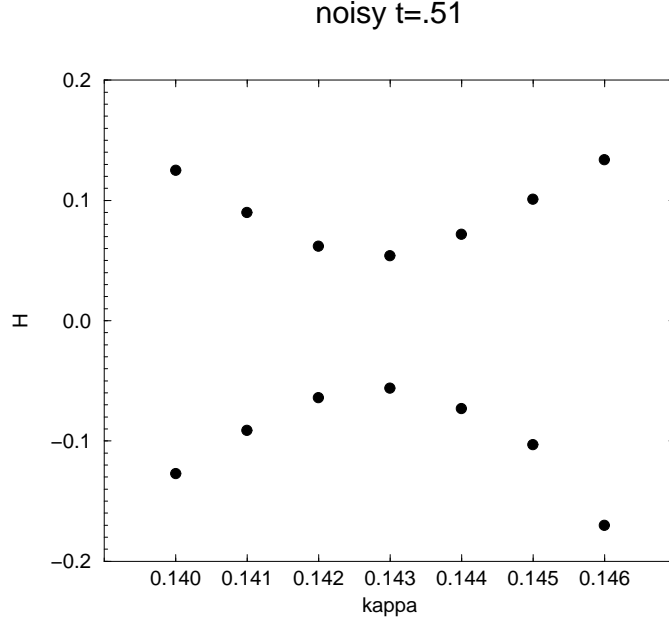


Figure 6. Third cross-section through double cone; avoided crossing.

cone degeneracy for the smooth configurations on lattices of size $L = 10$ and $L = 12$ and found them again at $t = .5$ (this point has a special symmetry) but at $\kappa = .1355(5)$ and $\kappa = .1320(5)$ respectively. Thus, as $L \rightarrow \infty$, κ approaches the expected value $.125$ at a rate proportional to $\frac{1}{L^2}$. The sign change in $\det(1 + V)$ will occur for any κ that exceeds those values, at the appropriate size L .

I conclude that WA is reproduced and that it is likely that the effect will be preserved by configurations that are not too “rough”.

Before closing, I would like to make a few related comments:

(a) If κ_0 happens to be very close to the degeneracy point we see that $\det \frac{1+V}{2}$ will not be defined everywhere along AB; instead of going through an order 2 zero, we would go through two exceptional configurations, one corresponding to the birth of an “instanton” and the subsequent one to its annihilation. A similar sequence would be observed in the vicinity of the tip of the double cone along a line where the gauge configuration is fixed and only κ is changing: as such the configuration might be interpreted as consisting of an instanton and an antiinstanton, and would seem similar to configurations studied in [13]. Obviously, only for quite smooth gauge configurations and well separated lumps of non-abelian field strength square and topological charge density does the latter interpretation become meaningful and these criteria do not apply here.

(b) The double-cone degeneracies we find in the real hamiltonian require the tuning of two real parameters and thus are of the generic type. This is what makes the phenomenon

robust. The reality of H is crucial.

(c) Suppose that, following [14,15], we wish to work with only the combination $V = \gamma_5 \epsilon(H)$ but not H itself. V , unlike H , because of instantons, must be nonanalytic in the gauge fields [10]. Thus, I see no easy way to include the Berry phase mechanism as an explanation of the sign change.

It may seem then that we would not even be able to understand why the zeros required for WA occur in $\det \frac{1+V}{2}$ in a *generic* way (in the sense that they require tuning of one real parameter, not two). But, this is not true: to understand the situation requires a more detailed analysis of the zeros of $1 + V$ than provided either in [10] or in [15]. Let me sketch what is needed below:

Assuming $\dim(\text{Ker}(H)) = 0$, define Z as the subspace of the space on which H acts as follows* :

$$Z = \text{Ker}([\gamma_5, \epsilon(H)]). \quad (3)$$

One easily proves the following result:

$$Z = Z_+ \oplus Z_-, \quad (4)$$

where

$$Z_{\pm} = \text{Ker}(1 \pm V). \quad (5)$$

Z contains all the eigenvectors of V with real eigenvalues. Following [10] we know that all complex eigenvalues of V are paired, so $\dim(Z)$ is even. $\dim(Z_-) + \dim(Z_+)$ can change only by even numbers. If $\dim(Z_-)$ changes by an odd amount (simplest case: by unity) this change must be matched by a jump in $\dim(Z_+)$. Such a change is non-analytical since no motion of a complex conjugate pair of eigenvalues can smoothly yield a pair of ± 1 eigenvalues. These “transitions” under variations of the background take us through exceptional configurations where V is ill defined. The entries of the matrix V are not analytic functions of the gauge background. Note that at the same time that we produce a zero eigenvector for $1 + V$ we also produce one for $1 - V$, the field dependent factor of the chiral transformation suggested in [17]. (Starting from [4], this transformation can be obtained as a remnant of the natural chiral symmetry fundamental to the overlap.)

Both spaces Z_{\pm} are invariant under γ_5 . Thus γ_5 can be diagonalized in each. The trace of γ_5 restricted to Z_- gives the index and is exactly the deficit/surplus in the filling

* In the more general version of the overlap, the role of γ_5 is played by $\epsilon(H')$ where $H' \neq 0$ and $\text{tr} \epsilon(H') \equiv 0$, for all gauge fields. More explicitly, H' is chosen of the same form as H , only $\kappa < \frac{1}{2d}$ [4].

of the ground state of \mathcal{H} in [4]. If this trace is non-vanishing, obviously $\dim(Z_-) > 0$, and the fermion determinant vanishes. Note that $\det \frac{1-V}{2}$ also vanishes when the trace of γ_5 restricted to Z_+ is non-zero. The trace of γ_5 when restricted to Z , always vanishes, since the trace of γ_5 when restricted only to states with complex eigenvalues of V vanishes because of the pairing.

Even if topology is trivial in the overlap sense, there still can be states in Z_- , only their total number must be even, because the trace of γ_5 in the subspace has to be zero. A pair of such states appears at the points where $\det \frac{1+V}{2}$ changes sign. Only one parameter would need to be fine tuned to attain such a transition point rather than the generic two: Close to this point, V , when restricted to the two crossing states, is well approximated by $-1 + A$, where A is real and antisymmetric; a two by two antisymmetric matrix depends on only one real parameter. We learn that as long as V exactly satisfies the Ginsparg-Wilson relation (GW) [14], we could recover all that really matters (even without knowing about Berry's phase). This is not surprising in view of the close connection between the overlap and GW emphasized in [16]. However, there is a serious danger that seemingly minor truncations of V would destroy the generic nature of the type of level crossings needed for the WA anomaly, making the levels avoid rather than cross.

If all we allowed ourselves to consider were V , because of the pairing of eigenvalues, it would be tempting to adopt a definition of the chiral determinant as the product of all $1 + \lambda$ where λ are the eigenvalues of V with positive imaginary part. As long as the “fixed” points ± 1 are avoided, the definition seems fine, even if not evidently local, at least smooth in the background. What we showed in this paper implies that the “fixed” points can't be avoided and such a definition is questionable. On the other hand, if we start with some fiducial gauge background, and decide to follow the above “half of eigenvalues” by analyticity in the gauge background, we lose single-valuedness in addition to gauge invariance. The loss of single-valuedness becomes evident when we investigate a closed loop in gauge configuration space made out of the path A to B we defined above followed by a path B to A induced by a smooth deformation of g to unity. The existence of this closed loop is a lattice phenomenon, and so is the associated lack of single-valuedness. Actually, by considering the deformations making up the paths AB and BA simultaneously, another disk necessarily containing a degeneracy point of H is defined and it would be interesting to test this numerically, since this disk is at constant κ .

(d) The zeros of $\det(1+V)$ related to WA occur at zero topology and require the tuning of one real parameter. Their role is distinct from the zeros associated with instantons and they may be relevant for the spontaneous breakdown of chiral symmetries. These configurations might be interpreted sometimes as instanton-antiinstantons, but, in this

view the configurations are less distinguished than when seen as necessary reflections of WA. It would be useful to see these zeros by numerically studying $1 + V$ directly. For the gauge configurations studied here we know where to look for these zeros: for example, in the smooth case, on an 8^4 lattice, they should occur for $t = .5$ and any $\kappa > .143$.

(e) Any continuum argument based on topology uses gauge backgrounds that are atypical in the path integral in the sense that they are too smooth. Such smooth configurations have most likely zero probability in the path integral. By taking the effect to the lattice we open the way to directly check whether the “noisy” backgrounds which are typical to the path integral preserve the features found assuming smoothness. All continuum considerations I am aware of have weathered very well this “latticization”. Still, I feel that at a strict level, something conceptual is being gained by going to the lattice in the way done here.

(f) The Wigner-Brillouin phase choice (WB) [4] will introduce new exceptional configurations where it is ill defined. The existence of these new exceptional points is necessary since the phase choice attempts to defined a section in some twisted Z_2 -bundles over certain submanifolds in the space of parameters of the Hamiltonians H . One would need to know where these new exceptional points are to determine to what extent the WB phase choice reproduces the WA. In [7] we saw that the WB phase choice often does reproduce the required global anomaly, but not always.

(g) The gauge transformation g can be deformed to unity on the lattice. For the WA to work correctly a WB exceptional point is needed also on the path connecting B to A by g -deformation to accommodate the needed jump in sign.

(h) Nontrivial anomaly cancelation in this case could be seen for example by taking a single Weyl multiplet with $I = \frac{3}{2}$ [9]. There should be no odd numbers of double-cone degeneracies of the type seen above. Note that unlike in the case of nontrivial cancelation of perturbative anomalies [8], there is no Berry curvature over the space of gauge configurations that needs to be eliminated before a “perfect” (in the sense of [4], unrelated to [15]) regularizations of the chiral gauge theory in question should become possible.

Another potential example of a chiral theory with similar properties would be a model containing one $I = \frac{1}{2}$ Weyl fermion and another $I = \frac{5}{2}$ Weyl fermion.

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References.

- [1] E. Witten, Phys. Lett. B117 (1982) 324.
- [2] S. Elitzur, V. Nair, Nucl. Phys. B243 (1984) 205; F. R. Klinkhamer, Phys. Lett. B256 (1991) 41.
- [3] P. Nelson, L. Alvarez-Gaume, Comm. Math. Phys. 99 (1985) 103.
- [4] R. Narayanan, H. Neuberger, Nucl. Phys. B 443 (1995) 305.
- [5] S. Randjbar-Daemi, J. Strathdee, Phys. Lett. B348 (1995) 543.
- [6] D. Kaplan, M. Schmaltz, Phys.Lett. B368 (1996) 44.
- [7] Y. Kikukawa, H. Neuberger, Nucl.Phys. B513 (1998) 735.
- [8] H. Neuberger, hep-lat/9802033.
- [9] H. Neuberger, hep-lat/9803011.
- [10] H. Neuberger, Phys.Lett. B417 (1998) 141.
- [11] G. Herzberg, H. C. Longuet-Higgins, Disc. Farad. Soc. 35 (1963) 77. For more details see the original contribution by M. V. Berry titled “The Quantum Phase, Five Years After”, in the book “Geometric Phases in Physics” by A. Shapere and F. Wilczek, (Worl Scientific, 1989).
- [12] B. Bunk, K. Jansen, M. Lüscher, H. Simma, DESY-Report(Sept. 94); T. Kalkreuter, H. Simma, Comp. Phys. Comm. 93 (1996) 33.
- [13] R. Edwards, U. Heller, R. Narayanan, hep-lat/9801015.
- [14] P. Ginsparg, K. Wilson, Phys. Rev. D25 (1982) 2649.
- [15] P. Hasenfratz, V. Laliena, F. Niedermayer, hep-lat/9801021.
- [16] R. Narayanan, hep-lat/9802018.
- [17] M. Lüscher, hep-lat/9802011.
- [18] I. Dasgupta, A. R. Levi, V. Lubicz, C. Rebbi, Comp. Phys. Comm. 98 (1996) 365.